HYBRID MOMENTS OF THE RIEMANN ZETA-FUNCTION

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Abstract. The "hybrid" moments

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^k \left(\int_{t-G}^{t+G} |\zeta(\tfrac{1}{2}+ix)|^\ell \,\mathrm{d}x\right)^m \mathrm{d}t$$

of the Riemann zeta-function $\zeta(s)$ on the critical line $\Re\mathrm{e}\, s = \frac{1}{2}$ are studied. The expected upper bound for the above expression is $O_\varepsilon(T^{1+\varepsilon}G^m)$. This is shown to be true for certain specific values of $k, \ell, m \in \mathbb{N}$, and the explicitly determined range of $G = G(T; k, \ell, m)$. The application to a mean square bound for the Mellin transform function of $|\zeta(\frac{1}{2}+ix)|^4$ is given.

1. Introduction

Power moments represent one of the most important parts of the theory of the Riemann zeta-function $\zeta(s)$, defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (\sigma = \Re\mathrm{e}\, s > 1),$$

and otherwise by analytic continuation. Of particular significance are the moments on the "critical line" $\sigma = \frac{1}{2}$, and a large literature exists on this subject (see e.g., the monographs [8], [9], [23], [24] and [26]). Let us define

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k} \,\mathrm{d}t, \tag{1.1}$$

where $k \in \mathbb{R}$ is a fixed, positive number. Naturally one would want to find an asymptotic formula for $I_k(T)$ for a given k, but this is an extremely difficult problem. Except when $k = 1$ and $k = 2$, no asymptotic formula for $I_k(T)$ is

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known yet, although there are plausible conjectures for such formulas (see e.g., the work of B. Conrey et al. [2]). In the absence of asymptotic formulas for $I_k(T)$, one would like then to obtain good upper bounds for $I_k(T)$. A simple bound for $|\zeta(\frac12+it)|^k$ is (see [9, Theorem 1.2] and [24])

$$|\zeta(\tfrac12+it)|^k \ll \log t\int_{t-1}^{t+1}|\zeta(\tfrac12+ix)|^k\,\mathrm{d}x+1, \tag{1.2}$$

where $k\in\mathbb{N}$ is fixed. The use of (1.2) allows one to replace a power of $|\zeta(\frac12+it)|$ by its integral over a suitable (short) interval. In employing this procedure one obviously loses something, but on the other hand one gains flexibility from the fact that explicit upper bound for $I_k(T+G)-I_k(T-G)$ are known only in the case when $k=1$ (see Lemma 1) and $k=2$ (see [9, Theorem 5.2] and [23]). In this way bounds for $I_{k+m\ell}(T)$ are reduced to the so-called "hybrid" moments of the type

$$\int_T^{2T}|\zeta(\tfrac12+it)|^k\left(\int_{t-G}^{t+G}|\zeta(\tfrac12+ix)|^\ell\,\mathrm{d}x\right)^m\mathrm{d}t\qquad(k,\ell,m\in\mathbb{N}), \tag{1.3}$$

where k,ℓ,m are assumed to be fixed, and $1\ll G\ll T$. The expected bound for the expression in (1.3) (this is consistent with the hitherto unproved Lindelöf hypothesis that $\zeta(\frac12+it)\ll_\varepsilon |t|^\varepsilon$) is clearly

$$O_\varepsilon(T^{1+\varepsilon}G^m). \tag{1.4}$$

Here and later ε (>0) denotes arbitrarily small constants, not necessarily the same ones at each occurrence, and $a=O_\varepsilon(b)$ (same as $a\ll_\varepsilon b$) means that the implied constant depends only on ε. The problem is to find, for given k,ℓ,m, the range of

$$G=G(T;k,\ell,m)$$

for which the integral (1.3) is bounded by (1.4), and naturally one would like the lower bound for G to be as small as possible. Note that from general results (e.g., see K. Ramachandra's monograph [24]) one obtains that the expression in (1.3) is, for $\log\log T\ll G\ll T$,

$$\gg G^m(\log T)^{\ell^2m/4}\int_T^{2T}|\zeta(\tfrac12+it)|^k\,\mathrm{d}t \gg TG^m(\log T)^{(\ell^2m+k^2)/4}. \tag{1.5}$$

This shows that, up to 'ε', the bound in (1.4) is indeed best possible. The (less difficult) case $k=0$ in (1.3) was investigated by the author in [13] and [14]. In particular, the latter work contains a proof of the bound

$$\int_T^{2T} J_2^m(t,G)\,\mathrm{d}t \ll_\varepsilon T^{1+\varepsilon} \tag{1.6}$$

for $T^{1/2+\varepsilon} \le G \le T$ if $m = 1, 2$; for $T^{4/7+\varepsilon} \le G \le T$ if $m = 3$, and for $T^{3/5+\varepsilon} \le G \le T$ if $m = 4$, where

$$J_k(T,G) := \frac{1}{\sqrt{\pi}G}\int_{-\infty}^{\infty} |\zeta(\tfrac12+iT+iu)|^{2k}\mathrm{e}^{-(u/G)^2}\,\mathrm{d}u \quad (k>0, 1 \ll G \ll T). \tag{1.7}$$

The bound (1.6) in the above range was obtained in [14] by employing Y. Motohashi's explicit formula (e.g., see [9] and [23]) for $J_2(t,G)$, which contains quantities from the spectral theory of the non-Euclidean Laplacian.

As for the applications of bounds for (1.3), note that the case (this is $k = \ell = 4, m = 1$) of the hybrid integral

$$\int_T^{2T} |\zeta(\tfrac12+it)|^4 \int_{t-G}^{t+G} |\zeta(\tfrac12+ix)|^4\,\mathrm{d}x\,\mathrm{d}t \tag{1.8}$$

appeared in [20] in connection with mean square bounds for the Mellin transform function, defined initially by

$$\mathcal{Z}_2(s) = \int_1^\infty |\zeta(\tfrac12+ix)|^4 x^{-s}\,\mathrm{d}x \qquad (\sigma>1), \tag{1.9}$$

and otherwise by analytic continuation. The functions $\mathcal{Z}_k(s)$ (in the general case $|\zeta(\frac12+ix)|^4$ is replaced by $|\zeta(\frac12+ix)|^{2k}$ for $\sigma > \sigma(k)\,(>1)$ with suitable $\sigma(k)$) are of great importance in the theory of power moments of $\zeta(\frac12+it)$ (see e.g., [11], [20]). It was shown by the author in [15] that

$$\int_1^T |\mathcal{Z}_2(\sigma+it)|^2\,\mathrm{d}t \ll_\varepsilon T^{\frac{15-12\sigma}{5}+\varepsilon} \qquad (\tfrac56 \le \sigma \le \tfrac54), \tag{1.10}$$

which is the sharpest bound for the range in question.

We shall obtain results on the integral in (1.3) when k, ℓ equal 2 or 4, which is logical, since it is in these cases that we have good information on $I_k(T)$. Namely let, for $k \in \mathbb{N}$ fixed,

$$I_k(T) = \int_0^T |\zeta(\tfrac12+it)|^{2k}\,\mathrm{d}t = T\,P_{k^2}(\log T) + E_k(T), \tag{1.11}$$

where for some suitable coefficients $a_{j,k}$ one has

$$P_{k^2}(y) = \sum_{j=0}^{k^2} a_{j,k} y^j, \tag{1.12}$$

and $E_k(T)$ is to be considered as the error term in (1.11). An extensive literature exists on $E_k(T)$, especially on $E_1(T) \equiv E(T)$ (see F.V. Atkinson's classical paper [1]), and the reader is referred to [9] for a comprehensive account. It is known that ($\gamma = -\Gamma'(1) = 0.5772157\ldots$ is Euler's constant)

$$P_1(y) = y + 2\gamma - 1 - \log(2\pi),$$

and $P_4(y)$ is a quartic polynomial in y whose leading coefficient equals $1/(2\pi^2)$. This was obtained in A.E. Ingham's classical work [7]. For an explicit evaluation of all the coefficients of $P_4(y)$ see e.g., the author's paper [10]. One hopes that

$$E_k(T) \;=\; o(T) \qquad (T \to \infty) \tag{1.13}$$

will hold for each fixed integer $k \ge 1$, which implies the Lindelöf hypothesis that $\zeta(\frac{1}{2} + it) \ll_\varepsilon |t|^\varepsilon$. So far (1.13) is known to be true only in the cases $k = 1$ and $k = 2$, when $E_k(T)$ is a true error term in the asymptotic formula (1.11). In particular we have (see [8],[9], [17], [18], [23]) $E(T) \ll_\varepsilon T^{\theta+\varepsilon}$ for some θ satisfying $\frac{1}{4} \le \theta < \frac{1}{3}$, and $E(T) = \Omega_\pm(T^{1/4})$. We also have $E_2(T) = \Omega_\pm(\sqrt{T})$ and the bounds (op. cit.)

$$E_2(T) \ll T^{2/3}\log^8 T, \qquad \int_0^T E_2^2(t)\,\mathrm{d}t \ll T^2 \log^{22} T. \tag{1.14}$$

As usual, $f(x) = \Omega_\pm(g(x))$ for a given $g(x)$ (> 0 for $x > x_0 > 0$) means that

$$\limsup_{x\to\infty} f(x)/g(x) > 0, \qquad \liminf_{x\to\infty} f(x)/g(x) < 0.$$

2. Statement of results

Before we state explicitly our results note that we have the bounds

$$\int_{T-G}^{T+G} |\zeta(\tfrac{1}{2} + it)|^2\,\mathrm{d}t \;\ll\; G\log T \qquad (T^{1/3} \ll G = G(T) \ll T), \tag{2.1}$$

and

$$\int_{T-G}^{T+G} |\zeta(\tfrac{1}{2} + it)|^4\,\mathrm{d}t \;\ll_\varepsilon\; GT^\varepsilon \qquad (T^{2/3} \ll G = G(T) \ll T). \tag{2.2}$$

This easily follows from estimates on $E(T)$ and $E_2(T)$ mentioned at the end of the last section. It means that we can restrict ourselves to the range $G \ll T^{1/3}$ when $\ell = 2$ in (1.3), and to the range $G \ll T^{2/3}$ when $\ell = 4$. This will be implicitly assumed in the proofs of our results, which are contained in

THEOREM 1. *We have*

$$\int_T^{2T} |\zeta(\tfrac12+it)|^2 \int_{t-G}^{t+G} |\zeta(\tfrac12+ix)|^2 \,\mathrm{d}x \;\mathrm{d}t \ll TG\log^2 T \quad \Big(T^\varepsilon \ll G = G(T) \ll T\Big), \tag{2.3}$$

$$\int_T^{2T} |\zeta(\tfrac12+it)|^2 \int_{t-G}^{t+G} |\zeta(\tfrac12+ix)|^4 \,\mathrm{d}x \;\mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}G \quad (T^{3/10} \ll G = G(T) \ll T), \tag{2.4}$$

(2.5)
$$\int_T^{2T} |\zeta(\tfrac12+it)|^2 \left(\int_{t-G}^{t+G} |\zeta(\tfrac12+ix)|^2 \,\mathrm{d}x\right)^2 \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}G^2 \; (T^{\frac17} \ll G = G(T) \ll T),$$

(2.6)
$$\int_T^{2T} |\zeta(\tfrac12+it)|^2 \left(\int_{t-G}^{t+G} |\zeta(\tfrac12+ix)|^2 \,\mathrm{d}x\right)^3 \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}G^3 \quad (T^{\frac15} \ll G = G(T) \ll T),$$

and for $T^{\frac{7}{12}} \log^C T \ll G = G(T) \ll T$ *we have*

$$\int_T^{2T} |\zeta(\tfrac12+it)|^2 \left(\int_{t-G}^{t+G} |\zeta(\tfrac12+ix)|^4 \,\mathrm{d}x\right)^2 \mathrm{d}t \;\ll\; TG^2 \log^{12} T. \tag{2.7}$$

THEOREM 2. *We have, for* $1 \ll G = G(T) \ll T$ *and some* $C > 0$,

$$\int_T^{2T} |\zeta(\tfrac12+it)|^4 \int_{t-G}^{t+G} |\zeta(\tfrac12+ix)|^4 \,\mathrm{d}x \;\mathrm{d}t \ll \log^C T\Big(TG + \min(T^{5/3}, T^2G^{-1})\Big), \tag{2.8}$$

(2.9)
$$\int_T^{2T} |\zeta(\tfrac12+it)|^4 \left(\int_{t-G}^{t+G} |\zeta(\tfrac12+ix)|^2 \,\mathrm{d}x\right)^2 \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}G^2 \; (T^{\frac{7}{24}} \le G = G(T) \ll T),$$

(2.10)
$$\int_T^{2T}|\zeta(\tfrac12+it)|^4\left(\int_{t-G}^{t+G}|\zeta(\tfrac12+ix)|^4\,{\rm d}x\right)^2\,{\rm d}t \ll_\varepsilon T^{1+\varepsilon}G^2\ (T^{\frac59}\le G=G(T)\ll T).$$

To assess the strength of our results note, for example, that (1.2) and (2.8) give

$$\begin{aligned}\int_T^{2T}|\zeta(\tfrac12+it)|^8\,{\rm d}t &\ll \log T\int_T^{2T}|\zeta(\tfrac12+it)|^4\int_{t-G}^{t+G}|\zeta(\tfrac12+ix)|^4\,{\rm d}x\;{\rm d}t\\ &+\int_T^{2T}|\zeta(\tfrac12+it)|^4\,{\rm d}t \ll T^{3/2}\log^C T\qquad (G=T^{1/2}).\end{aligned}\tag{2.11}$$

The bound in (2.11), which follows easily by the Cauchy-Schwarz inequality for integrals from estimates of the fourth and twelfth moment of $|\zeta(\frac12+it)|$ (see [4] and [8, Chapter 8]), is the strongest known bound for the eighth moment of $|\zeta(\frac12+it)|$.

Our last result concerns an improvement of (1.10). Let ρ be such a constant for which

$$\int_0^T|\zeta(\tfrac12+it)|^8\,{\rm d}t \ \ll_\varepsilon\ T^{\rho+\varepsilon}\tag{2.12}$$

holds. At present we have $1\le\rho\le 3/2$. The lower bound follows from general principles (see [8, Chapter 9]). The upper bound is a consequence of (2.11), and its improvements would be very significant. We shall prove, using (2.4), the following

THEOREM 3. *If* $\mathcal{Z}_2(s)$ *is defined by* (1.9) *and* ρ *is defined by* (2.12), *then*

$$\int_1^T|\mathcal{Z}_2(\sigma+it)|^2\,{\rm d}t \ \ll_\varepsilon\ T^{\frac{4\rho+4-8\sigma}{3\rho-1}+\varepsilon}\qquad \Bigl(\frac{5+\rho}{8}\le\sigma\le\frac{1+\rho}{2}\Bigr).\tag{2.13}$$

Corollary. We have

$$\begin{aligned}\int_1^T|\mathcal{Z}_2(\tfrac{13}{16}+it)|^2\,{\rm d}t \ &\ll_\varepsilon\ T^{1+\varepsilon}\quad \Bigl(\frac{13}{16}=0.8125\Bigr),\\ \int_1^T|\mathcal{Z}_2(1+it)|^2\,{\rm d}t \ &\ll_\varepsilon\ T^{4/7+\varepsilon}\quad \Bigl(\frac47=0.571428\ldots\Bigr).\end{aligned}\tag{2.14}$$

Note that (1.10) gives

$$\int_1^T|\mathcal{Z}_2(\tfrac56+it)|^2\,{\rm d}t\ll_\varepsilon T^{1+\varepsilon},\quad \int_1^T|\mathcal{Z}_2(1+it)|^2\,{\rm d}t\ll_\varepsilon T^{3/5+\varepsilon},$$

while (2.14) improves both of these bounds, since $13/16<5/6$ and $4/7<3/5$.

3. The necessary lemmas

In this section we shall state some lemmas that are necessary for the proofs of our theorems. The first is an explicit formula for an integral involving $|\zeta(\frac12+it)|^2$.

LEMMA 1. *For $T^\varepsilon \le G \le T^{1-\varepsilon}$ we have*

$$
\begin{aligned}
&\frac{1}{\sqrt{\pi}G}\int_{-\infty}^{\infty}|\zeta(\tfrac12+iT+iy)|^2\mathrm{e}^{-(y/G)^2}\,\mathrm{d}y = O(\log T) + \\
&+\sqrt{2}\sum_{n=1}^{\infty}(-1)^n d(n)n^{-1/2}\left(\left(\frac{T}{2\pi n}+\frac14\right)^{1/2}-\frac12\right)^{-1/2}\times \\
&\times\exp\left(-G^2\left(\operatorname{arsinh}\sqrt{\frac{\pi n}{2T}}\right)^2\right)\sin f(T,n),
\end{aligned}
\tag{3.1}
$$

where $d(n)$ is the number of divisors of n, $\operatorname{ar\,sinh} z = \log(z+\sqrt{z^2+1}\,)$, and

$$
\begin{aligned}
f(T,n) &= 2T\operatorname{arsinh}\left(\sqrt{\frac{\pi n}{2T}}\right)+\sqrt{2\pi nT+\pi^2n^2}-\tfrac14\pi \\
&= -\tfrac14\pi+2\sqrt{2\pi nT}+\tfrac16\sqrt{2\pi^3}n^{3/2}T^{-1/2}+a_5n^{5/2}T^{-3/2}+a_7n^{7/2}T^{-5/2}+\dots
\end{aligned}
\tag{3.2}
$$

for $1\le n \ll T$, where a_{2m-1} are suitable constants.

Proof of Lemma 1. The proof of (3.1) (see also [14]) is based on Y. Motohashi's exact formula [23, Theorem 4.1]. It states that

$$
\begin{aligned}
&\int_{-\infty}^{\infty}|\zeta(\tfrac12+it)|^2 g(t)\,\mathrm{d}t = \int_{-\infty}^{\infty}\left[\Re\mathrm{e}\left\{\frac{\Gamma'}{\Gamma}(\tfrac12+it)\right\}+2\gamma-\log(2\pi)\right]g(t)\,\mathrm{d}t \\
&+2\pi\Re\mathrm{e}\,(g(\tfrac12 i))+4\sum_{n=1}^{\infty}d(n)\int_0^{\infty}(y(y+1))^{-1/2}g_c(\log(1+1/y))\cos(2\pi ny)\,\mathrm{d}y,
\end{aligned}
\tag{3.3}
$$

where

$$g_c(x) := \int_{-\infty}^{\infty} g(t)\cos(xt)\,\mathrm{d}t$$

is the cosine Fourier transform of $g(t)$. One requires the function $g(r)$ in (3.3) to be real-valued for $r\in\mathbb{R}$, and that there exists a large constant $A>0$ such that $g(r)$ is regular and $\ll (|r|+1)^{-A}$ for $|\Im\mathrm{m}\, r|\le A$. The choice

$$g(t) = \frac{1}{\sqrt{\pi}G}\mathrm{e}^{-(T-t)^2/G^2},\quad g_c(x) = \mathrm{e}^{-\frac14(Gx)^2}\cos(Tx)$$

is permissible, and then the integral on the left-hand side of (3.3) becomes (see (1.7)) $J_1(T,G)$. The first integral on the right-hand side of (3.3) is $O(\log T)$, and the second one is evaluated by the saddle-point method (see e.g., [8, Chapter 2]). A convenient result to use is [8, Theorem 2.2 and Lemma 15.1], due originally to Atkinson [1] for the evaluation of exponential integrals $\int_a^b \varphi(x)\exp(2\pi iF(x))\,dx$. In the latter only the exponential factor $\exp(-\frac14 G^2\log(1+1/y))$ is missing. In the notation of [1] and [8] we have that the saddle point x_0 (root of $F'(x)=0$) satisfies

$$x_0 = U - \frac12 = \left(\frac{T}{2\pi n}+\frac14\right)^{1/2} - \frac12,$$

and the presence of the above exponential factor makes it possible to truncate the series in (3.3) at $n = TG^{-2}\log T$ with a negligible error. Furthermore, in the remaining range for n we have (in the notation of [8])

$$\Phi_0\mu_0F_0^{-3/2} \ll (nT)^{-3/4},$$

which makes a total contribution of $O(1)$, as does error term integral in Theorem 2.2 of [8]. The error terms with $\Phi(a)$, $\Phi(b)$ vanish for $a\to 0+$, $b\to+\infty$, and (3.1) follows. Finally note that by using Taylor's formula it is seen that the error made by replacing

$$\left(\left(\frac{T}{2\pi n}+\frac14\right)^{1/2}-\frac12\right)^{-1/2}\exp\left(-G^2\left(\operatorname{arsinh}\sqrt{\frac{\pi n}{2T}}\right)^2\right)$$

with

$$\left(\frac{T}{2\pi n}\right)^{-1/4}\exp\Bigl(-\frac{\pi G^2 n}{2T}\Bigr)$$

in (3.1) is $\ll 1$ for $G \ge T^{1/5}\log^C T$.

LEMMA 2. *If $A\in\mathbb{R}$ is a constant, then we have*

(3.4)
$$\cos\left(\sqrt{8\pi nT}+\tfrac16\sqrt{2\pi^3}n^{3/2}T^{-1/2}+A\right) = \int_{-\infty}^{\infty}\alpha(u)\cos(\sqrt{8\pi n}(\sqrt{T}+u)+A)\,du,$$

where $\alpha(u)\ll T^{1/6}$ for $u\ne 0$,

(3.5)
$$\alpha(u)\ll T^{1/6}\exp(-bT^{1/4}|u|^{3/2})$$

for $u<0$, and

(3.6)
$$\alpha(u) = T^{1/8}u^{-1/4}\left(d\exp(ibT^{1/4}u^{3/2})+\bar d\exp(-ibT^{1/4}u^{3/2})\right)+O(T^{-1/8}u^{-7/4})$$

for $u \ge T^{-1/6}$ *and some constants* $b\ (>0)$ *and* d.

This is a result of M. Jutila [22, Part II]. The formulas (3.5)–(3.6) enable us to get rid of the term $n^{3/2}T^{-1/2}$ in $f(T,n)$ in Lemma 1. By using the Taylor expansion, or a variation of the above method of Jutila, the influence of the remaining terms in $f(T,n)$ can be made innocuous as well in our applications.

LEMMA 3. *Let* $\mathcal{N}$ *denote the number of solutions in integers m,n,k of the inequality*

$$|\sqrt{m}+\sqrt{n}-\sqrt{k}\,| \ \le\ \delta\sqrt{M} \qquad (\delta > 0)$$

with $M' < n \le 2M', M < m \le 2M, k \in \mathbb{N}$, *and* $M' \le M$. *Then*

$$\mathcal{N} \ \ll_\varepsilon\ M^\varepsilon\Big(M^2M'\delta + (MM')^{1/2}\Big). \tag{3.7}$$

LEMMA 4. *Let* $k \ge 2$ *be a fixed integer and* $\delta > 0$ *be given. Then the number of integers* n_1,n_2,n_3,n_4 *such that* $N < n_1,n_2,n_3,n_4 \le 2N$ *and*

$$|n_1^{1/k}+n_2^{1/k}-n_3^{1/k}-n_4^{1/k}| \ <\ \delta N^{1/k}$$

is, for any given $\varepsilon > 0$,

$$\ll_\varepsilon\ N^\varepsilon(N^4\delta + N^2). \tag{3.8}$$

Lemma 3 was proved by Sargos and the author [19], while Lemma 4 is due to Robert–Sargos [25]. They represent powerful arithmetic tools which are essential in the analysis when the cube or biquadrate of exponential sums involving $\sqrt{n}$ appears.

LEMMA 5. *For* $HU \gg T^{1+\varepsilon}$ *and* $T^\varepsilon \ll U \le \frac12\sqrt{T}$ *we have*

$$\begin{aligned}&\int_T^{T+H}\Big(E(x+U)-E(x)\Big)^2\,\mathrm{d}x\\ &= \frac{1}{4\pi^2}\sum_{n\le\frac{T}{2U}}\frac{d^2(n)}{n^{3/2}}\int_T^{T+H}x^{1/2}\left|\exp\left(2\pi iU\sqrt{\frac{n}{x}}\right)-1\right|^2\mathrm{d}x + O_\varepsilon(T^{1+\varepsilon}+HU^{1/2}T^\varepsilon).\end{aligned} \tag{3.9}$$

This result was proved by M. Jutila [21]. The analogous formula also holds with $E(T)$ replaced by

$$\Delta(x) \ :=\ \sum_{n\le x} d(n) - x(\log x + 2\gamma - 1), \tag{3.10}$$

the error term in the classical Dirichlet divisor problem. From (3.9) Jutila deduced ($a \asymp b$ means that $a \ll b \ll a$)

$$\int_T^{2T}\Bigl(E(x+U)-E(x)\Bigr)^2\,\mathrm{d}x \asymp TU\log^3\Bigl(\frac{\sqrt{T}}{U}\Bigr) \qquad \Bigl(T^\varepsilon \ll U \le \tfrac12\sqrt{T}\Bigr). \tag{3.11}$$

The author recently sharpened (3.11) to an asymptotic formula. Namely it was proved in [16] that, with suitable constants e_j ($e_3 > 0$) and $T^\varepsilon \ll U \le \frac12\sqrt{T}$,

$$\begin{aligned}\int_T^{2T}\Bigl(E(x+U)-E(x)\Bigr)^2\,\mathrm{d}x &= TU\sum_{j=0}^{3} e_j \log^j\Bigl(\frac{\sqrt{T}}{U}\Bigr)\\ &\quad + O_\varepsilon(T^{1/2+\varepsilon}U^2) + O_\varepsilon(T^{1+\varepsilon}U^{1/2}).\end{aligned}$$

4. The proof of Theorem 1

We begin with the bound in (2.3). The left-hand side equals, by the defining relation of $E(T)$ ((1.10)–(1.11) with $k=1$),

$$\begin{aligned}&\int_T^{2T}|\zeta(\tfrac12+it)|^2\Bigl(O(G\log T)+E(t+G)-E(t-G)\Bigr)\,\mathrm{d}t\\ &\ll GT\log^2 T + \int_T^{2T}|\zeta(\tfrac12+it)|^2|E(t+G)-E(t-G)|\,\mathrm{d}t\\ &\ll GT\log^2 T + \left(\int_T^{2T}|\zeta(\tfrac12+it)|^4\,\mathrm{d}t\int_T^{2T}\Bigl(E(t+G)-E(t-G)\Bigr)^2\,\mathrm{d}t\right)^{1/2}\\ &\ll GT\log^2 T + (T\log^4 T\cdot TG\log^3 T)^{1/2} \ll GT\log^2 T\end{aligned}$$

for $G \ge T^\varepsilon$, as asserted. Here we used the Cauchy-Schwarz inequality for integrals and (3.11). Note that the upper bound in (2.3) is best possible, as it coincides with the lower bound in (1.5). An interesting, but difficult problem, would be to obtain an asymptotic formula for the integral in (2.3).

To discuss (2.4), we first exchange the order of integration in the relevant integrals. It follows that the left-hand side of (2.4) does not exceed

$$\begin{aligned}&\int_{T-G}^{2T+G}|\zeta(\tfrac12+ix)|^4\left(\int_{x-G}^{x+G}|\zeta(\tfrac12+it)|^2\,\mathrm{d}t\right)\mathrm{d}x\\ &=\int_{T-G}^{2T+G}|\zeta(\tfrac12+ix)|^4\Bigl(O(G\log T)+E(x+G)-E(x-G)\Bigr)\,\mathrm{d}x\\ &\ll_\varepsilon GT\log^5 T + T^{547/416+\varepsilon},\end{aligned}\tag{4.1}$$

which immediately gives the bound which is somewhat weaker than the one in (2.4), since $13/10 = 1.3 < 547/416 = 0.314903\ldots$. Here we used the sharpest known bound $E(T) \ll_\varepsilon T^{131/416+\varepsilon}$, $131/416 = 0.314903\ldots$ of N. Watt (see [5] and [6]). To obtain the sharper bound asserted by (2.4) we shall use results on the moments of $E^*(t)$ (see Section 5), and hence the proof of the bound in question will be completed there.

For the proof of (2.5) we start from (1.2) which gives, for $T/2 \le t \le 5T/2$,

$$|\zeta(\tfrac{1}{2}+it)|^2 \ll \log T \int_{t-T^\varepsilon}^{t+T^\varepsilon} |\zeta(\tfrac{1}{2}+ix)|^2 \,\mathrm{d}x + 1,$$

and we use the trivial inequality

$$\begin{aligned}&\int_{t-G}^{t+G} |\zeta(\tfrac{1}{2}+ix)|^k \,\mathrm{d}x = \int_{-G}^{G} |\zeta(\tfrac{1}{2}+it+iu)|^k \,\mathrm{d}u\\ &\le \mathrm{e}\int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2}+it+iu)|^k \mathrm{e}^{-(u/G)^2} \,\mathrm{d}u = \sqrt{\pi}\mathrm{e}GJ_k(t,G)\end{aligned}$$

in the notation of (1.7), where $T/2 \le t \le 5T/2, 1 \ll G \ll T$ and $k \in \mathbb{N}$ is fixed. This gives

$$\begin{aligned}&\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^2 \left(\int_{t-G}^{t+G} |\zeta(\tfrac{1}{2}+ix)|^2 \,\mathrm{d}x\right)^2 \mathrm{d}t\\ &\ll T^\varepsilon \log T \int_{T/2}^{5T/2} \varphi(t) J_1(t,T^\varepsilon) \left(\int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2}+it+iu)|^2 \mathrm{e}^{-(u/G)^2} \,\mathrm{d}u\right)^2 \mathrm{d}t\\ &+ G^2 T \log^4 T,\end{aligned}$$

following the proof of (2.3), where $\varphi(t)\,(\ge 0)$ is a smooth function supported in $[T/2,\, 5T/2]$, such that $\varphi(t) = 1$ for $T \le t \le 2T$ and $\varphi^{(r)}(t) \ll_r T^{-r}$ for $t \in \mathbb{R}$ and any $r \in \mathbb{N}$. For $J_1(t, T^\varepsilon)$ we use Lemma 1, writing $\sin z = (\mathrm{e}^{iz} - \mathrm{e}^{-iz})/(2i)$, and integrate by parts $\exp(i2\sqrt{2\pi n t}\,)$. In this way it is seen that

$$\begin{aligned}&\int_{T/2}^{5T/2} \varphi(t) \sum_{n=1}^{\infty} (-1)^n d(n) n^{-1/2} \ldots \exp\bigl(if(t,n)\bigr) \left(\int_{-\infty}^{\infty} \ldots\right)^2 \mathrm{d}t\\ &= \frac{i}{2\sqrt{\pi n}} \int_{T/2}^{5T/2} \Biggl\{ \varphi(t)\sqrt{t} \sum_{n=1}^{\infty} (-1)^n d(n) n^{-1/2} \ldots\\ &\times \exp\Bigl(-i\tfrac{1}{4}\pi + i\tfrac{1}{6}\sqrt{2\pi^3} n^{3/2} t^{-1/2} + a_5 i n^{5/2} t^{-3/2} + \ldots\Bigr) \left(\int_{-\infty}^{\infty} \ldots\right)^2 \Biggr\}' \,\mathrm{d}t.\end{aligned}$$

Note that, for $G = T^\varepsilon, t \asymp T$, we have

$$
\begin{aligned}
&\left\{\exp\left(-G^2\left(\operatorname{arsinh}\sqrt{\frac{\pi n}{2t}}\right)^2\right)\right\}' \\
&= \frac{G^2\sqrt{\frac{\pi n}{2}}\operatorname{arsinh}\sqrt{\frac{\pi n}{2t}}}{t^{3/2}\sqrt{1+\frac{\pi n}{2t}}}\exp\left(-G^2\left(\operatorname{arsinh}\sqrt{\frac{\pi n}{2t}}\right)^2\right) \\
&\asymp \frac{G^2 n}{t^2}\exp\left(-G^2\left(\operatorname{arsinh}\sqrt{\frac{\pi n}{2t}}\right)^2\right),
\end{aligned}
\tag{4.2}
$$

$$(\varphi(t)\sqrt{t})' \ll \frac{1}{\sqrt{T}}\,,$$

and that

$$
\begin{aligned}
&\left\{\left(\int_{-\infty}^{\infty}|\zeta(\tfrac{1}{2}+it+iu)|^2 \mathrm{e}^{-(u/G)^2}\,\mathrm{d}u\right)^2\right\}' \\
&= 2\int_{-\infty}^{\infty}|\zeta(\tfrac{1}{2}+it+iu)|^2 \mathrm{e}^{-(u/G)^2}\,\mathrm{d}u\int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}|\zeta(\tfrac{1}{2}+it+iu)|^2 \mathrm{e}^{-(u/G)^2}\,\mathrm{d}u.
\end{aligned}
$$

Integrating by parts we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\Big\{\frac{\mathrm{d}}{\mathrm{d}t}|\zeta(\tfrac{1}{2}+it+iu)|^2\Big\}\mathrm{e}^{-(u/G)^2}\,\mathrm{d}u \\
&= \int_{-\infty}^{\infty}\Big\{\frac{\mathrm{d}}{\mathrm{d}u}|\zeta(\tfrac{1}{2}+it+iu)|^2\Big\}\mathrm{e}^{-(u/G)^2}\,\mathrm{d}u \\
&= 2\int_{-\infty}^{\infty}uG^{-2}|\zeta(\tfrac{1}{2}+it+iu)|^2 \mathrm{e}^{-(u/G)^2}\,\mathrm{d}u.
\end{aligned}
\tag{4.3}
$$

Observe the integrals in (4.3) can be truncated at $|u| = G\log T$ with a negligible error. Therefore, after an integration by parts, we get an integral with the same type of exponential factor (i.e., $f(t,n)$ in the exponential), but there will be in the integrand a smooth factor of the order $\ll G^{-1}\sqrt{T/n}$. Hence after a large number of integrations by parts it follows that the contribution of n satisfying $n > T^{1+\varepsilon}G^{-2}$ will be negligible (i.e., less than T^{-A} for any given $A > 0$ and $\varepsilon = \varepsilon(A)$). This truncation of the series over n is the crucial point in the proof, as the ensuing expression will be quite similar to the expressions for $J_1(t,G)$, only in the exponential factor in (4.2) we shall have $G = T^\varepsilon$. Thus the proof reduces to the estimation of

$$T^\varepsilon G^2\int_{T/2}^{5T/2}\varphi(t)\sum\nolimits_1\Big(\sum\nolimits_2\Big)^2\,\mathrm{d}t, \tag{4.4}$$

where

$$\sum_2 := \sum_{n\le T^{1+\varepsilon}G^{-2}} (-1)^n d(n) n^{-1/2}\left(\left(\frac{t}{2\pi n}+\frac14\right)^{1/2}-\frac12\right)^{-1/2}\times$$
$$\times \exp\left(-G^2\left(\operatorname{arsinh}\sqrt{\pi n/(2t)}\right)^2\right)\sin f(t,n),$$

and $\sum_1$ is the same expression with $G = T^\varepsilon$ in the exponential factor. The other two terms, which arise after the squaring of the right-hand side of (3.1), are clearly less difficult to deal with. Note that

$$\sum_1\left(\sum_2\right)^2 = \sum_{m\le T^{1+\varepsilon}G^{-2}} (-1)^m d(m)\cdots \sin f(t,m)\times$$
$$\sum_{n\le T^{1+\varepsilon}G^{-2}} (-1)^n d(n)\cdots \sin f(t,n) \sum_{k\le T^{1+\varepsilon}G^{-2}} (-1)^k d(k)\cdots \sin f(t,k),$$

and write the sines as exponentials. We use Lemma 2 and its variants to remove the terms $a_3 n^{3/2}t^{-1/2}+\ldots$ from the exponentials, in the same way as was explained in detail in the author's works [12]–[14]. Then we integrate by parts many times, as was done in the previous part of the proof. It transpires that the non-negligible contribution will come from the triplets (m,n,k) $(\in \mathbb{N}^3)$ satisfying

$$|\sqrt{m}+\sqrt{n}-\sqrt{k}\,| \le T^{\varepsilon-1/2}\ (M'<n\le 2M', M<m\le 2M, K\le k\le 2K), \tag{4.5}$$

where $M'\le M$ and $K\asymp M$. Namely the terms with

$$\sqrt{m}+\sqrt{n}+\sqrt{k},\quad -\sqrt{m}-\sqrt{n}-\sqrt{k}$$

are clearly negligible, so only the combination of signs as in (4.5) is relevant. Furthermore, by the first derivative test ([8, Lemma 2.1]) it is seen that the contribution is small if $K < AM$ or $K > BM$ with suitable positive constants A, B. The contribution of the triplets (m,n,k) satisfying (4.5) is estimated by (3.7) of Lemma 3 with $\delta = T^{\varepsilon-1/2}M^{-1/2}$. The total contribution is then, by trivial estimation of the remaining portion of the integral in (4.4),

$$\begin{aligned}
&\ll_\varepsilon T^{1+\varepsilon}G^2 \max_{K,M,M'\le T^{1+\varepsilon}G^{-2},K\asymp M} T^{-3/4}M^{-3/4}(M^{5/2}T^{-1/2}+M) + T^{1+\varepsilon}G^2\\
&\ll_\varepsilon T^{1+\varepsilon}G^2 \max_{K,M,M'\le T^{1+\varepsilon}G^{-2},K\asymp M}(T^{-5/4}M^{7/4}+T^{-3/4}M^{1/4}) + T^{1+\varepsilon}G^2\\
&\ll_\varepsilon T^{1+\varepsilon}G^2(T^{1/2}G^{-7/2}+1) \ll_\varepsilon T^{1+\varepsilon}G^2
\end{aligned}$$

for $G\ge T^{1/7}$, as asserted.

The proof of (2.6) is similar to the proof of (2.5). The major difference is that, instead of (4.4), now we shall have to bound

$$T^\varepsilon G^3\int_{T/2}^{5T/2}\varphi(t)\sum\nolimits_1\Bigl(\sum\nolimits_2\Bigr)^3\,{\rm d}t. \tag{4.6}$$

We use then Hölder's inequality to deduce that the integral in (4.6) does not exceed

$$\left(\int_{T/2}^{5T/2}\varphi(t)\Bigl|\sum\nolimits_1\Bigr|^4{\rm d}t\right)^{1/4}\left(\int_{T/2}^{5T/2}\varphi(t)\Bigl|\sum\nolimits_2\Bigr|^4{\rm d}t\right)^{3/4}. \tag{4.7}$$

Both integrals in (4.7) are estimated similarly. The sum over n in $\sum_1$ is split into $O(\log T)$ subsums where $N < n \le N' \le 2N$, with $N \ll T^{1+\varepsilon}G^{-2}$. We develop the biquadrates and integrate many times by parts, as in the previous case. Instead of Lemma 3 we use Lemma 4, since only the quadruples (k,ℓ,m,n) $(\in \mathbb{N}^4)$ such that

$$|\sqrt{m}+\sqrt{n}-\sqrt{k}-\sqrt{\ell}\,| \;<\; T^{\varepsilon-1/2}$$

will make a non-negligible contribution. The number of such quadruples is estimated by (3.8) (with $\delta = T^{\varepsilon-1/2}N^{-1/2}$), and afterwards the relevant integral is estimated trivially. The total contribution will be

$$\begin{aligned}&\ll_\varepsilon T^{1+\varepsilon}G^3\max_{N\ll T^{1+\varepsilon}G^{-2}}T^{-1}N^{-1}(N^4T^{\varepsilon-1/2}N^{-1/2}+N^2)+T^{1+\varepsilon}G^3\\&\ll_\varepsilon T^{1+\varepsilon}G^3(TG^{-5}+1)\ll_\varepsilon T^{1+\varepsilon}G^3\end{aligned}$$

for $G \ge T^{1/5}$, as asserted. It can be readily checked that the contribution of other terms on the right-hand side of (3.1), which arise from cubing the relevant expression, is also $\ll_\varepsilon T^{1+\varepsilon}G^3$ for $G \ge T^{1/5}$.

To prove (2.7), note that the left-hand side equals (see (1.11) with $k=2$)

$$\begin{aligned}&\int_T^{2T}|\zeta(\tfrac12+it)|^2\Bigl(O(G\log^4T)+E_2(t+G)-E_2(t-G)\Bigr)^2{\rm d}t\\&\ll TG^2\log^{12}T+\Bigl(\int_T^{2T}|\zeta(\tfrac12+it)|^4\,{\rm d}t\int_{T/3}^{3T}E_2^4(t)\,{\rm d}t\Bigr)^{1/2}\\&\ll TG^2\log^{12}T+T^{13/6}\log^CT\ll TG^2\log^{12}T\end{aligned} \tag{4.8}$$

for $G \ge T^{7/12}\log^C T$, as asserted. Here we used the bound, which follows from (1.14), namely

$$\int_0^T|E_2(t)|^A\,{\rm d}t \ll T^{2+\frac23(A-2)}\log^CT \quad (A\ge2,\ C=22+8(A-2)) \tag{4.9}$$

with $A = 4$. This completes the proof of Theorem 1.

5. The proof of Theorem 2

We have first, similarly as in (4.8),

$$
\begin{aligned}
&\int_T^{2T} |\zeta(\tfrac12+it)|^4 \int_{t-G}^{t+G} |\zeta(\tfrac12+ix)|^4 \,\mathrm{d}x\ \mathrm{d}t\\
&= \int_T^{2T} |\zeta(\tfrac12+it)|^4 \Big(O(G\log^4 T) + E_2(t+G) - E_2(t-G)\Big)\,\mathrm{d}t \qquad (5.1)\\
&\ll TG\log^8 T + T^{5/3}\log^C T,
\end{aligned}
$$

where we used the first bound in (1.14). This implies that the left-hand side of (5.1) is $\ll (TG+T^{5/3})\log^C T$ in the whole range $1 \ll G = G(T) \ll T$. The bound in question was actually proved by Ivić-Jutila-Motohashi [20] in connection with mean square estimates for $\mathcal{Z}_2(s)$. Hence the main task is to prove the other bound in (2.8), for which we need the second bound in (1.14). Note that the left-hand side of (2.8) is majorized by a multiple ($L = \log T$) of

$$
\begin{aligned}
&\int_T^{2T} |\zeta(\tfrac12+it)|^4 \int_{-\infty}^{\infty} |\zeta(\tfrac12+it+iu)|^4 \mathrm{e}^{-(u/G)^2}\,\mathrm{d}u\,\mathrm{d}t\\
&\ll TGL^8 + \int_T^{2T} |\zeta(\tfrac12+it)|^4 \int_{-\infty}^{\infty} \left\{\frac{\mathrm{d}}{\mathrm{d}t}E_2(t+u)\right\}\mathrm{e}^{-(u/G)^2}\,\mathrm{d}u\,\mathrm{d}t\\
&\ll TGL^8 + 2\int_T^{2T} |\zeta(\tfrac12+it)|^4 \int_{-\infty}^{\infty} E_2(t+u)uG^{-2}\mathrm{e}^{-(u/G)^2}\,\mathrm{d}u\,\mathrm{d}t \qquad (5.2)\\
&\ll TGL^8 + \frac{L}{G}\int_{T-GL}^{2T+GL} |E_2(u)| \int_{u-GL}^{u+GL} |\zeta(\tfrac12+it)|^4\,\mathrm{d}t\,\mathrm{d}u\\
&\ll L^C\left\{TG + T^{3/2} + G^{-2}\int_{T-GL}^{2T+GL} |E_2(u)| \int_{u-GL^2}^{u+GL^2} |E_2(t)|\,\mathrm{d}t\,\mathrm{d}u\right\}.
\end{aligned}
$$

Here we used the fact that

$$
\frac{\mathrm{d}}{\mathrm{d}t}E_2(t+u) = \frac{\mathrm{d}}{\mathrm{d}u}E_2(t+u), \qquad (5.3)
$$

and integrated by parts. We used a similar procedure later, namely ($Q_4 = P_4 + P_4'$ (see (1.11)–(1.12)) is a suitable polynomial of degree four, C is henceforth a generic

positive constant, $u \asymp T$)

$$
\begin{aligned}
&\int_{u-GL}^{u+GL} |\zeta(\tfrac{1}{2}+it)|^4 \,\mathrm{d}t \ll \int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2}+iu+iv)|^4 \mathrm{e}^{-(v/GL)^2} \,\mathrm{d}v \\
&= \int_{-\infty}^{\infty} \Bigl(Q_4(\log(u+v)) + E_2'(u+v)\Bigr)\mathrm{e}^{-(v/GL)^2} \,\mathrm{d}v \\
&= O(GL^C) + 2\int_{-\infty}^{\infty} vG^{-2}E_2(u+v)\mathrm{e}^{-(v/GL)^2} \,\mathrm{d}v \\
&\ll L^C\Bigl(G + G^{-1}\int_{u-GL^2}^{u+GL^2} |E_2(t)| \,\mathrm{d}t\Bigr).
\end{aligned}
$$

We also used the bound, which follows by the Cauchy-Schwarz inequality for integrals from the mean square bound in (1.14), namely

$$
\int_0^T |E_2(t)| \,\mathrm{d}t \;\ll\; T^{3/2}\log^C T.
$$

To complete the proof it remains to note that

$$
\begin{aligned}
&\left(\int_{T-GL}^{2T+GL} |E_2(u)| \int_{u-GL^2}^{u+GL^2} |E_2(t)| \,\mathrm{d}t \,\mathrm{d}u\right)^2 \\
&\ll \int_{T-GL}^{2T+GL} E_2^2(u) \,\mathrm{d}u \int_{T-GL}^{2T+GL} \Bigl(\int_{u-GL^2}^{u+GL^2} |E_2(t)| \,\mathrm{d}t\Bigr)^2 \mathrm{d}u \\
&\ll T^2L^C \int_{T-GL}^{2T+GL} GL^2 \int_{u-GL^2}^{u+GL^2} E_2^2(t) \,\mathrm{d}t \,\mathrm{d}u \\
&\ll T^2GL^C \int_{T-GT^\varepsilon}^{2T+T^\varepsilon} E_2^2(t) \left(\int_{t-GL^2}^{t+GL^2} \mathrm{d}u\right) \mathrm{d}t \\
&\ll T^4L^CG^2.
\end{aligned}
\tag{5.4}
$$

When we take the square root in (5.4) and insert the resulting bound in (5.2) we are left with the bound

$$
O\Bigl(L^C(TG + T^{3/2} + T^2G^{-1})\Bigr)
$$

for the left-hand side of (2.8). But as $T^{3/2} \le TG$ for $G \ge T^{1/2}$ and $T^{3/2} \le T^2G^{-1}$ for $G \le T^{1/2}$, this means that the term $T^{3/2}$ may be omitted, and (2.8) follows. We point out yet another estimate, namely (6.2), for the integral in (2.8). This was derived for the proof of Theorem 3, and does not contain the (expected) term $TG\log^C T$, but terms which are reasonably small when G is 'about' $T^{131/416}$.

The proof of (2.9) is based on the use of (5.3) and the fourth moment of the function $E^*(t)$, defined by

$$E^*(t) \;:=\; E(t) - 2\pi\Delta^*\bigl(\frac{t}{2\pi}\bigr),$$

where (see (3.10))

$$\Delta^*(x) := -\Delta(x) + 2\Delta(2x) - \tfrac{1}{2}\Delta(4x) = \tfrac{1}{2}\sum_{n\le 4x}(-1)^n d(n) - x(\log x + 2\gamma - 1),$$

and $\Delta(x)$ is the error term in the Dirichlet divisor problem. The function $E^*(t)$ was investigated by several authors, including M. Jutila [22], who introduced it, and the author [12]–[14]. Among other things, the author ([12, Part II) proved that

$$\int_0^T |E^*(t)|^5 \,\mathrm{d}t \;\ll_\varepsilon\; T^{2+\varepsilon} \tag{5.5}$$

and that ([12, Part IV, Corollary 2])

$$\int_0^T \bigl(E^*(t)\bigr)^4 \,\mathrm{d}t \;\ll_\varepsilon\; T^{7/4+\varepsilon}. \tag{5.6}$$

The advantage of working with $E^*(t)$ instead of $E(t)$ is that the former is, in the mean power sense, smaller than the latter (for this see [8, Chapter 15] and [12]).

For our proof we need from [13] the elementary formula (cf. (3.4))

$$J_1(t,G) = \frac{2}{\sqrt{\pi}G^3}\int_{-G\log T}^{G\log T} xE^*(t+x)\mathrm{e}^{-(x/G)^2}\,\mathrm{d}x + O(\log^2 T), \tag{5.7}$$

which is valid for $T^\varepsilon \le G = G(T) \le T^{1/3}$, $T/2 \le t \le 5T/2$. First observe that the left-hand side of (2.9) is majorized ($\varphi(t)$ is as in the proof of (2.5)) by

$$\begin{aligned}
&G^2\int_{T/2}^{5T/2} |\zeta(\tfrac{1}{2}+it)|^4\varphi(t)J_1^2(t,G)\,\mathrm{d}t\\
&= G^2\int_{T/2}^{5T/2} (Q_4(\log t) + E_2'(t))\varphi(t)J_1^2(t,G)\,\mathrm{d}t\\
&= O(TG^2\log^8 T) - G^2\int_{T/2}^{5T/2} E_2(t)\Bigl(\varphi(t)J_1^2(t,G)\Bigr)'\,\mathrm{d}t.
\end{aligned} \tag{5.8}$$

Namely by the Cauchy-Schwarz inequality for integrals and the classical bound

$$\int_0^T |\zeta(\tfrac12+it)|^4\,{\rm d}t \ll T\log^4 T,$$

we have

$$\begin{aligned}
&G^2\int_{T/2}^{5T/2} Q_4(\log t)\varphi(t)J_1^2(t,G)\,{\rm d}t\\
&\ll G\log^4 T\int_{T/2}^{5T/2}\int_{-\infty}^{\infty}|\zeta(\tfrac12+it+iu)|^4{\rm e}^{-(u/G)^2}\,{\rm d}u\,{\rm d}t\\
&\ll G\log^4 T\int_{T/3}^{3T}|\zeta(\tfrac12+ix)|^4\int_{x-GL}^{x+GL}{\rm e}^{-(x-t)^2/G^2}\,{\rm d}t\,{\rm d}x \ll TG^2\log^8 T.
\end{aligned}$$

We also use $\varphi'(t)\ll 1/T$ (the contribution of this derivative is easily handled) and (5.7). Hence with suitable $g(t)$ we obtain
(5.9)

$$\Bigl(J_1^2(t,G)\Bigr)' = 2J_1(t,G)\Bigl\{-\frac{2}{\sqrt{\pi}G^3}\int\limits_{-G\log T}^{G\log T}E^*(t+x)\bigl(1-\frac{2x^2}{G^2}\bigr){\rm e}^{-(x/G)^2}\,{\rm d}x + g'(t)\Bigr\},$$

where $g(t)\ll\log^2 T$. The term $g'(t)$ is integrated back by parts, and its contribution is easily seen to be $\ll_\varepsilon T^{1+\varepsilon}G^2$ in the required range. The first term on the right-hand side of (5.9) is inserted in (5.8). It follows that the main contribution to the left-hand side of (5.8) is ($L=\log T$) bounded by $T^{1+\varepsilon}G^2$ plus

$$\begin{aligned}
&G^2\int_{T/2}^{5T/2}|E_2(t)|\varphi(t)G^{-5}\Bigl(\int_{-GL}^{GL}|E^*(t+x)|\,{\rm d}x\Bigr)^2{\rm d}t\\
&\ll G^{-3}\Bigl(\int_{T/2}^{5T/2}|E_2(t)|^2\,{\rm d}t\Bigr)^{1/2}\Bigl(\int_{T/2}^{5T/2}\Bigl(\int_{-GL}^{GL}|E^*(t+x)|\,{\rm d}x\Bigr)^4{\rm d}t\Bigr)^{1/2}\\
&\ll TL^CG^{-3}\left(\int_{T/2}^{5T/2}G^3\int_{t-GL}^{t+GL}|E^*(u)|^4\,{\rm d}u\,{\rm d}t\right)^{1/2}\\
&\ll TL^CG^{-3}\left(\int_{T/2-GL}^{5T/2+GL}G^3|E^*(u)|^4\Bigl(\int_{u-GL}^{u+GL}{\rm d}t\Bigr)\,{\rm d}u\right)^{1/2}\\
&\ll_\varepsilon T^{15/8+\varepsilon}G^{-1},
\end{aligned}$$

where we used Hölder's inequality for integrals, (5.3) and (5.6). Since

$$T^{15/8}G^{-1}\le TG^2\qquad(\text{for } G\ge T^{7/24}),$$

the bound in (2.9) follows.

Finally it remains to prove the bound (2.10) of Theorem 2. We proceed similarly as in the proof of (2.9), and we majorize first $\int_T^{2T} \ldots$ by $\int_{T/2}^{5T/2} \varphi(t) \ldots$. Then we majorize

$$\left(\int_{t-G}^{t+G} |\zeta(\tfrac12+ix)|^4 \,\mathrm{d}x\right)^2$$

by $G^2 J_2^2(t,G)$ and write

$$|\zeta(\tfrac12+it)|^4 = Q_4(\log t) + E_2'(t), |\zeta(\tfrac12+it+iu)|^4 = Q_4(\log(t+u)) + E_2'(t+u).$$

After this we integrate by parts E_2', using (5.3) and obtaining $E_2(t), E_2(t+u)$, but gaining essentially a factor of $1/G$ each time in the process. The major contribution to the left-hand side of (2.10) will be $\ll_\varepsilon T^{1+\varepsilon}G^2$ plus
(5.10)

$$G^{-3}\int_{T/2}^{5T/2} \varphi(t)|E_2(t)|\left(\int_{-GL}^{GL} |E_2(t+x)|^4 \mathrm{e}^{-(x/G)^2}\,\mathrm{d}x\right)^2 \mathrm{d}t$$

$$\ll G^{-3}\Bigl(\int_{T/2}^{5T/2} \varphi(t)|E_2(t)|^2\,\mathrm{d}t\Bigr)^{1/2}\left(\int_{T/2}^{5T/2}\Bigl(\int_{t-GL}^{t+GL} |E_2(u)|\,\mathrm{d}u\Bigr)^4 \mathrm{d}t\right)^{1/2}.$$

Now we use (4.9) with $A = 4$ and Hölder's inequality, to obtain that

$$\begin{aligned}
&\int_{T/2}^{5T/2}\left(\int_{t-GL}^{t+GL} |E_2(u)|\,\mathrm{d}u\right)^4 \mathrm{d}t \\
&\ll\ (GL)^3\int_{T/2}^{5T/2}\int_{t-GL}^{t+GL} |E_2(u)|^4\,\mathrm{d}u \\
&=\ GL^3\int_{T/2-GL}^{5T/2+GL} E_2^4(u)\left(\int_{u-GL}^{u+GL} \mathrm{d}t\right)\mathrm{d}u \\
&\ll\ T^{10/3}G^4\log^C T.
\end{aligned} \tag{5.11}$$

Hence if we insert (5.11) in (5.10) and use (5.3), we obtain that the expression in (5.10) is

$$\ll\ T^{8/3}G^{-1}\log^C T\ \ll_\varepsilon\ T^{1+\varepsilon}G^2$$

for $G \ge T^{5/9}$, which yields then (2.10) and completes the proof of Theorem 2.

It remains yet to complete the proof of (2.4). From (4.1) we have

$$
\begin{aligned}
&\int_{T-G}^{2T+G} |\zeta(\tfrac12+ix)|^4 \left(\int_{x-G}^{x+G} |\zeta(\tfrac12+it)|^2\,\mathrm{d}t\right)\mathrm{d}x\\
&= \int_{T-G}^{2T+G} |\zeta(\tfrac12+ix)|^4 \Bigl(O(G\log T) + E(x+G) - E(x-G)\Bigr)\,\mathrm{d}x \qquad (5.12)\\
&\ll_\varepsilon GT^\varepsilon + \int_{T-G}^{2T+G} |\zeta(\tfrac12+ix)|^4 \Bigl|E^*(x+G) - E^*(x-G)\Bigr|\,\mathrm{d}x.
\end{aligned}
$$

Here we used the defining property of $E^*(t)$ together with the elementary bound

$$\Delta^*(x+G) - \Delta^*(x-G) \ \ll_\varepsilon\ GT^\varepsilon \qquad (x \asymp T,\ 1 \ll G \ll T),$$

which is easily obtained, since

$$\Delta^*(x) = \tfrac12 \sum_{n\le 4x} (-1)^n d(n), \qquad d(n) \ll_\varepsilon n^\varepsilon.$$

At this point we use Hölder's inequality for integrals, (5.5) and the bound (see [8, Chapter 8])

$$\int_0^T |\zeta(\tfrac12+it)|^5\,\mathrm{d}t \ \ll_\varepsilon\ T^{9/8+\varepsilon},$$

to deduce that the last integral in (5.12) is

$$
\begin{aligned}
&\le \left(\int_{T/3}^{3T} |\zeta(\tfrac12+ix)|^5\,\mathrm{d}x\right)^{4/5} \left(\int_{T/3}^{3T} |E^*(x)|^5\,\mathrm{d}x\right)^{1/5}\\
&\ll_\varepsilon T^{\frac98\cdot\frac45+2\cdot\frac15+\varepsilon} = T^{\frac{13}{10}+\varepsilon},
\end{aligned}
$$

which yields (2.4). For small values of G, namely for $1 \ll G \ll T^{1/10}$, the bound in (2.4) may be improved by using a more general result than (3.11), namely

$$\int_T^{T+H} \Bigl(E(x+U) - E(x)\Bigr)^2\,\mathrm{d}x \ \asymp\ HU\log^3\left(\frac{\sqrt{T}}{U}\right), \qquad (5.13)$$

deduced by M. Jutila [21] from (3.9), for $HU \gg T^{1+\varepsilon}$ and $T^\varepsilon \ll U \le \frac12\sqrt{T}$. If (5.13) is applied in conjunction with (4.1), the Cauchy-Schwarz inequality and the bound (see (2.11)) $\int_0^T |\zeta(\frac12+it)|^8\,\mathrm{d}t \ll_\varepsilon T^{3/2+\varepsilon}$, we obtain (2.4) with $T^\varepsilon(GT + T^{5/4}G^{1/2})$, and

$$T^{5/4}G^{1/2} \ \le\ T^{13/10} \qquad (1 \ll G \le T^{1/10}).$$

This is unconditional, but conjectures on the order of $E(T)$ and $E^*(T)$ would lead to further improvements, e.g., the conjectural bound $E(T) \ll_\varepsilon T^{1/4+\varepsilon}$ would replace the exponent 13/10 in (2.4) by 5/4.

We end this section by pointing out that one can improve (1.6) for the range given in Section 1. Namely, since the integral $J_k(t, G)$ in (1.7) can be truncated at $u = \pm G \log T$ with an error which is $\ll T^{-A}$ for any given $A > 0$, it follows that (1.6) is equivalent to

$$\int_T^{2T} \Bigl(\int_{t-G}^{t+G} |\zeta(\tfrac12 + ix)|^4 \,\mathrm{d}x\Bigr)^m \,\mathrm{d}t \ll_\varepsilon T^{1+\varepsilon} G^m. \tag{5.14}$$

We proceed as in the proof of (2.7), using (4.9), to infer that the integral in (5.14) is (as before $C > 0$ is a generic constant)

$$\begin{aligned} &\ll TG^m (\log T)^{4m} + \int_{T/3}^{3T} |E_2(t)|^m \,\mathrm{d}t \\ &\ll TG^m (\log T)^{4m} + T^{2+\frac23(m-2)} \log^C T \\ &\ll TG^m (\log T)^C \end{aligned}$$

for

$$G \ \ge \ T^{\frac{2m-1}{3m}} \qquad (m \ge 2), \tag{5.15}$$

where incidentally m does not have to be an integer. In particular, it follows then from (5.15) that (1.6) holds for $G \ge T^{1/2}$ $(m = 2)$, $G \ge T^{5/9}$ $(m = 3)$ and $G \ge T^{7/12}$ $(m = 4)$. Since $5/9 < 4/7$ and $7/12 < 3/5$, this means that we have improved the range of G for which (1.6) holds when $m = 3, 4$.

6. The proof of Theorem 3

The method of obtaining mean square bounds for $\mathcal{Z}_2(s)$ (see (1.9)) was developed in [15] and [20], so that we shall be fairly brief. From [20, pp. 337-339], we have

(6.1)
$$\begin{aligned} &\int_T^{2T} |\mathcal{Z}_2(\sigma + it)|^2 \,\mathrm{d}t \ll_\varepsilon T^{2+\varepsilon} X^{1-2\sigma} + \\ &+ \log T \sup_{T^{1-\varepsilon} \le K \le X} \int_{T/2}^{5T/2} \int_K^{2K} |\zeta(\tfrac12 + ix)|^4 x^{-2\sigma} \int_{x-KT^{\varepsilon-1}}^{x+KT^{\varepsilon-1}} |\zeta(\tfrac12 + iy)|^4 \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}t \\ &\ll_\varepsilon T^{2+\varepsilon} X^{1-2\sigma} + T \log T \sup_{T^{1-\varepsilon} \le K \le X} K^{-2\sigma} \int_K^{2K} |\zeta(\tfrac12 + ix)|^4 \int_{x-G}^{x+G} |\zeta(\tfrac12 + iy)|^4 \,\mathrm{d}y \,\mathrm{d}x \end{aligned}$$

with $G = KT^{\varepsilon-1}$, $\sigma > 1/2$, and X a parameter to be suitably chosen. For the last integral above one could use (2.8) of Theorem 2. However, this would not lead to the result of Theorem 3, as we need a bound when G is 'about' $K^{1/3}$, or even slightly smaller. Thus we shall proceed differently, and use the elementary inequality

$$ab \le \tfrac{1}{2}(a^{1/2}b^{3/2} + a^{3/2}b^{1/2}) \qquad (a, b \ge 0)$$

to obtain

$$\begin{aligned}
&\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^4 \int_{t-G}^{t+G} |\zeta(\tfrac{1}{2}+ix)|^4 \,\mathrm{d}x\,\mathrm{d}t \\
&\ll \int_T^{2T}\int_{t-G}^{t+G} \Big(|\zeta(\tfrac{1}{2}+it)|^2|\zeta(\tfrac{1}{2}+ix)|^6 + |\zeta(\tfrac{1}{2}+it)|^6|\zeta(\tfrac{1}{2}+ix)|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\ll \int_{T-G}^{2T+G} |\zeta(\tfrac{1}{2}+ix)|^6 \int_{x-G}^{x+G} |\zeta(\tfrac{1}{2}+it)|^2 \,\mathrm{d}t\,\mathrm{d}x \\
&+ \int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^6 \int_{t-G}^{t+G} |\zeta(\tfrac{1}{2}+ix)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\ll \int_{T/3}^{3T} |\zeta(\tfrac{1}{2}+it)|^6 \Big(O(G\log T) + E(t+G) - E(t-G)\Big)\,\mathrm{d}t \\
&\ll_\varepsilon T^{\varepsilon+\frac{1+\rho}{2}}(G + T^{\frac{131}{416}}).
\end{aligned} \tag{6.2}$$

Here we used the defining relation of $E(T)$ together with the sharpest known bound $E(T) \ll_\varepsilon T^{131/416+\varepsilon}$ (see [5], [6]). We also used the bound for the sixth moment which follows from (2.12) and the Cauchy-Schwarz inequality, namely

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^6\,\mathrm{d}t \le \left(\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,\mathrm{d}t \int_0^T |\zeta(\tfrac{1}{2}+it)|^8\,\mathrm{d}t\right)^{1/2} \ll_\varepsilon T^{\frac{1+\rho}{2}+\varepsilon}.$$

We use (6.2) (with $G = KT^{\varepsilon-1}$) in (6.1) to obtain

$$\begin{aligned}
&T \sup_{T^{1-\varepsilon}\le K\le X} K^{-2\sigma} \int_K^{2K} |\zeta(\tfrac{1}{2}+ix)|^4 \int_{x-G}^{x+G} |\zeta(\tfrac{1}{2}+iy)|^4\,\mathrm{d}y\,\mathrm{d}x \\
&\ll_\varepsilon T^{1+\varepsilon} \sup_{T^{1-\varepsilon}\le K\le X} \Big(K^{\frac{1+\rho}{2}+\frac{131}{416}-2\sigma} + T^{-1}K^{\frac{3+\rho}{2}-2\sigma}\Big) \\
&\ll_\varepsilon T^{1+\varepsilon} + T^\varepsilon X^{\frac{3+\rho}{2}-2\sigma}
\end{aligned}$$

if

$$\frac{1+\rho}{4} + \frac{131}{832} \le \sigma \le \frac{3+\rho}{4}. \tag{6.3}$$

If (6.3) holds, then we obtain from (6.1)

$$
\begin{aligned}
\int_T^{2T} |\mathcal{Z}_2(\sigma+it)|^2\,\mathrm{d}t &\ll_\varepsilon T^\varepsilon(T + X^{\frac{3+\rho}{2}-2\sigma} + T^2X^{1-2\sigma}) \\
&\ll_\varepsilon T^\varepsilon(T + T^{\frac{6+2\rho-8\sigma}{\rho+1}}) \ll_\varepsilon T^{1+\varepsilon}
\end{aligned}
\tag{6.4}
$$

for $\sigma \ge (5+\rho)/8$ if we choose $X = T^{4/(1+\rho)}$. However, for $\rho \le 181/104 = 1.74038\ldots$ we have

$$\frac{1+\rho}{4} + \frac{131}{832} \le \frac{5+\rho}{8}.$$

With $\rho = 3/2$ we have $(5+\rho)/8 = 13/16$, and the first assertion of (2.14) follows. Note that from (see (2.12))

$$\int_X^{2X} |\zeta(\tfrac12 + ix)|^8 x^{1-2\sigma}\,\mathrm{d}x \ll 1 \qquad (\sigma > \tfrac12(1+\rho)),$$

one deduces easily that (see e.g., [11, Lemma 4])

$$\int_T^{2T} |\mathcal{Z}_2(\sigma+it)|^2\,\mathrm{d}t \ll 1 \qquad (\sigma > \tfrac12(1+\rho)). \tag{6.5}$$

Thus the bound in (2.13) follows from (6.4), (6.5) and the convexity of mean values for regular functions (cf. [8, Lemma 8.3]). Setting $\sigma = 1, \rho = 3/2$ in (2.13) we obtain the second bound in (2.14). We remark that, by [11, eq. (3.21)], we have

$$\int_T^{2T} |\zeta(\tfrac12 + it)|^8\,\mathrm{d}t \ll_\varepsilon T^{2\sigma-1}\int_0^{T^{1+\varepsilon}} |\mathcal{Z}_2(\sigma+it)|^2\,\mathrm{d}t \quad (\tfrac12 < \sigma \le 1). \tag{6.6}$$

The bound (6.6) links the eighth moment of $|\zeta(\frac12 + it)|$ to the mean square of $\mathcal{Z}_2(s)$. Thus the second bound in (2.14) gives the value $\rho = 11/7$, which is close to the best known bound $\rho = 3/2$.

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